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## Abstract

We show how Vanstone's construction, given in his paper "A note on a construction for BIBD's", Utilitas Mathematica, 7(1975), 321-322, can be applied to symmetric GBRD(v, k, λ; │G│). │G│ odd, can be used to obtain GBRD(v, (v2), (k2), λ, (λ2); G) and hence many families of BIBD.


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# VANSTONE'S CONSTRUCTION APPLIED TO BHASKAR RAO DESIGNS

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## ABSTRACT

We show how Vanstone's construction, given in his paper "A note on a construction for BIBD's", *Utilitas Mathematica*, 7(1975), 321-322, can be applied to symmetric GBRD(v, k, λ; | G | ), | G | odd, can be used to obtain GBRD(v, $\binom{v}{2}$, $\binom{k}{2}$, λ, $\binom{\lambda}{2}$ ;G) and hence many families of BIBD.

## 1. INTRODUCTION

Definitions of SBIBD and BIBD are standard.

Let $A = [a_{ij}]$ be a matrix of order with $a_{ij} \varepsilon \{0, 1, -1\}$. $A$ is called a *weighing matrix* of weight $p$ and order $n$, if $AA^T = A^TA = pI_n$, where $I_n$ denotes the identity matrix of order $n$. Such a matrix is denoted by $W(n, p)$. If squaring all its entries gives an incidence matrix of a SBIBD then $W$ is called a balanced weighing matrix.

An *Hadamard matrix*, $A = [a_{ij}]$, is a $W(n,n)$, that is, it is a square matrix of order $n$ with entries $a_{ij} \varepsilon \{1,-1\}$ which satisfies

$$AA^T = A^TA = nI_n.$$

A *generalized Hadamard matrix* $\mathrm{GH}(gh,G) = (g_{ij}) = H$ over the group $G$ of order $g$ is a $gh \times gh$ matrix such that

(i)$g_{ij} \varepsilon G$ for all $1 \le i,\ j \le gh$, and

(ii) $\sum_{k=1}^{gh} g_{ik} g_{jk}^{-1} = \sum_{a \varepsilon G} ha$ whenever $i \neq j$ where the summation is in the group ring

$R(G)$. We also write this as

$$HH^* = hG.$$

Suppose we have a matrix $W$ with elements from an elementary abelian group $G = [h_1, h_2, ..., h_g]$, where $W = h_1A_1 + h_2A_2 + \cdots + h_gA_g$; here $A_1, \ldots, A_g$ are v × b (0,1) matrices, and the Hadamard product $A_i * A_j$ $(i \neq j)$ is zero. Suppose $(a_{i1}, \ldots, a_{ib})$ and $(b_{j1}, \ldots, b_{jb})$ are the $i$th and $j$th rows of $W$; then we define $WW^*$ by

$$(WW^*)_{ij} = (a_{i1},...,a_{ib}) \cdot (b_{j1}^{-1},...,b_{jb}^{-1})$$

with · designating the scalar product. Then $W$ is a *generalized Bhaskar Rao design* or *GBRD* if

$$\text{(i)} \quad WW^* = rI + \sum_{i=1}^{m} (c_i G) B_i$$

$$\text{(ii)} \quad N = A_1 + \cdots + A_g \text{ satisfies } NN^T = rI + \sum_{i=1}^{m} \lambda_i B_i,$$

that is, $N$ is the incidence matrix of a *PBIBD*$(m)$, and $(c_i G)$ gives the number of times a complete copy of the group $G$ occurs.

Such a matrix will be denoted by $GBRD_G(\text{v,b,r,k};\lambda_1, \ldots, \lambda_m; c_1, \ldots, c_m)$. In this paper we shall only be concerned with $m = 1, c = \lambda/g$, and $B_1 = J - I$. In this case $N$ is the incidence matrix of a *PBIBD*(1), that is a *BIBD*. Hence, the equations become:

$$\text{(i)} \quad WW^+ = rI + \lambda G/g\ (J - I)$$

$$\text{(ii)} \quad NN^T = (r - \lambda)I + \lambda J.$$

Thus $W$ is a $GBRD_G(\text{v,b,r,k,}\lambda)$. Since $\lambda(\text{v}-1) = r(k-1)$ and $bk = \text{vr}$, we sometimes use the notation *GBRD* (v,k,λ;G).

## 2. THE CONSTRUCTION

In his 1975 paper, Vanstone gave a powerful method for constructing BIBD from SBIBDs. We show his method applies to symmetric GBRD over groups which have no elements of order 2.

**THEOREM 1.** *Suppose there is a symmetric* GBRD(v, k, λ; G), | G | *odd* , *then there is a* GBRD(v, $\binom{\text{v}}{2}$, $\binom{\text{k}}{2}$, λ, $\binom{\lambda}{2}$ ; G ).

*Proof:* We modify the construction Vanstone used to show that an SBIBD(v, k, λ) yields a BIBD(v, $\binom{\text{v}}{2}$, $\binom{\text{k}}{2}$, λ, $\binom{\lambda}{2}$).

Let $A = (a_{ij})$ be the incidence matrix of the GBRD(v, k, λ; G). Label the columns of a v × $\binom{\text{v}}{2}$ matrix $B = (b_{ij})$, with the $n$ = $\binom{\text{v}}{2}$ pairs from the set {1,...,v}.

Consider the column labelled $xy$, $(b_{1k}, \cdots, b_{vk})^T$, choose

$$b_{ik} = a_{ix}a_{iy}\ ,\ i = 1,...,v.$$

Clearly, every element of $B$ is zero or a group element, as that was true of $A$.

To establish the inner product property, we consider the inner product of two distinct rows

$$\sum_{k=1}^{n} b_{ik}b_{jk}^{-1} = \sum_{1 \le x < y \le n} a_{ix}a_{iy}a_{jy}^{-1}a_{ix}^{-1}\ ,\ i \ne j.$$

We first note that, for any group $G$ of order $g$ with elements $g_1, g_2, \cdots, g_g$

$$\begin{aligned} G^2 &= (g_1 + g_2 + \cdots + g_g)^2 \\ &= gG. \end{aligned}$$

With $(G + \cdots G)$ denoting t copies of G,

$$(G+G+...+G)^2 = tG^2 + 2\binom{t}{2}G^2 = t^2gG.$$

Since $g$ is odd and $n = v = tg$, if $g_1, \cdots, g_v$ are the elements of a row of the GBRD, $g_1^2,...,g_v^2 = tG$.

Hence, noting

$$(\sum x_i)^2 = \sum x_i^2 + 2\sum_{i \ne j} x_i x_j,$$

$$\begin{aligned} \sum_{1 \le x < y \le n} a_{ix}a_{iy}a_{jx}^{-1}a_{jy}^{-1} &= \tfrac{1}{2}\left[\sum_{k=1}^{n} a_{ik}a_{jk}^{-1}\right]^2 - \tfrac{1}{2}\sum_{k=1}^{n}(a_{ik}a_{jk}^{-1})^2 \\ &= \tfrac{1}{2}(G+G+...+G)^2 - \tfrac{1}{2}tG \qquad (t\ copies) \\ &= \tfrac{1}{2}(t^2g - t)G. \end{aligned}$$

Now, we know from Vanstone's result that a BIBD(v,k,λ) gives a BIBD(v, $\binom{v}{2}$, $\binom{k}{2}$, λ, $\binom{\lambda}{2}$). Thus, we wish to show a GBRD(v, k, λ; G) gives a GBRD(v, $\binom{v}{2}$, $\binom{k}{2}$, λ, $\binom{\lambda}{2}$ ; G ) . Certainly, the underlying BIBD has these parameters. The GBRD(v, k, λ; G) has $t = \lambda / g$ copies of the group as the inner product of each pair of rows and the constructed GBRD needs to have $\binom{\lambda}{2}$ / g copies of the group as the inner product of each pair of rows. But

$$\binom{\lambda}{2} / g = \tfrac{1}{2}\lambda(\lambda-1) / g = \tfrac{1}{2}t(tg-1)$$

as required. □

**Example 1.** Let the group of order 3, $\mathbf{Z}_3$, have generator ω. Represent ω by 1, $\omega^2$ by 2 and $\omega^3$ by 0. Then, the GH( 6,$\mathbf{Z}_3$) or GBRD( 6, 6, 6;$\mathbf{Z}_3$ ) is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix},$$

yielding

| 12 | 13 | 14 | 15 | 16 | 23 | 24 | 25 | 26 | 34 | 35 | 36 | 45 | 46 | 56 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 0 | 0 | 2 | 1 | 0 | 0 |
| 1 | 0 | 1 | 2 | 2 | 1 | 2 | 0 | 0 | 1 | 2 | 2 | 0 | 0 | 1 |
| 2 | 1 | 0 | 1 | 2 | 0 | 2 | 0 | 1 | 1 | 2 | 0 | 1 | 2 | 0 |
| 2 | 2 | 1 | 0 | 1 | 1 | 0 | 2 | 0 | 0 | 2 | 0 | 1 | 2 | 1 |
| 1 | 2 | 2 | 1 | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 2 | 0 | 2 | 1 |

a GBRD(6,15,15,6,15;$Z_3$).

**Example 2.** Proceed as in Example 1, but represent the zero element by *. Then the GBRD(5,4,3;$Z_3$)

| | | | | |
|---|---|---|---|---|
| * | 0 | 0 | 0 | 0 |
| 0 | * | 0 | 1 | 2 |
| 0 | 0 | * | 2 | 1 |
| 0 | 1 | 2 | * | 0 |
| 0 | 2 | 1 | 0 | * |

yields the GBRD(5,10,6,3,3;$Z_3$ ) :

| 12 | 13 | 14 | 15 | 23 | 24 | 25 | 34 | 35 | 45 |
|---|---|---|---|---|---|---|---|---|---|
| * | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 |
| * | 0 | 1 | 2 | * | * | * | 1 | 2 | 0 |
| 0 | * | 2 | 1 | * | 2 | 1 | * | * | 0 |
| 1 | 2 | * | 0 | 0 | * | 1 | * | 2 | * |
| 2 | 1 | 0 | * | 0 | 2 | * | 1 | * | * |

This method is so powerful when applied to generalized Hadamard matrices that we give it as a theorem in its own right.

## 3. USING GENERALIZED HADAMARD MATRICES IN THE CONSTRUCTION TO FORM BIBDS

**THEOREM 2.** *Suppose there is a* GH(tg;G), $|G| = g$ *odd. Then there is a* GBRD( tg, $\left[\frac{tg}{2}\right]$, $\left[\frac{tg}{2}\right]$, tg, $\left[\frac{tg}{2}\right]$ ; G ). *This can be used to form a*

$$\text{GDD}(\ g(tg+1),\ g\left[\tfrac{tg}{2}\right],\ \left[\tfrac{tg}{2}\right],\ tg+1,\ \lambda_1 = 0,\ \lambda_2 = \tfrac{1}{2}t(tg-1),\ m = g,\ n = tg+1).$$

**COMMENT.** The following construction is valid for any GH(2 | G | ; G) but these are presently only known for prime power orders | G | . The BIBD's constructed would be multiples of biplanes SBIBD($2p^2+p+1$, $2p+1$, 2) but these are not generally known as yet.

**THEOREM 3.** *Let p be any prime power. Then there exists a* BIBD($2p^2+p+1$, $p(2p^2+p+1)$, $p(2p+1)$, $2p+1$, $2p$).

*Proof.* We note a GH($2p$,$EA(p)$) exists for every prime power (Jungnickle (1979), D.J.Street (1979)). Use Theorem 2 to form a GBRD($2p$, $p(2p-1)$, $p(2p-1)$, $2p$, $p(2p-1)$; $EA(p)$). We replace each element of the GBRD by its $p\times p$ permutation matrix representation to obtain a (0,1) matrix $B$. Let $e$ be the $1\times p(2p-1)$ matrix of ones. Then

$$A=\begin{bmatrix} I_p\times e \\ B \end{bmatrix}$$

is a GDD($2p^2+p$, $p^2(2p-1)$, $p(2p-1)$, $2p+1$, $\lambda_1=0$, $\lambda_2=(2p-1)$).

Now a BIBD($2p+1$, $p(2p+1)$, $2p$, 2, 1) exists. Let $C$ be the matrix obtained from this BIBD by replacing each 1 and 0 in its incidence matrix by the $p\times 1$ matrices of ones and zeros respectively. Then the matrix

$$[C:A]$$

has $2p^2+p$ rows, $2p^3+p^2+p$ columns, $2p^2+p$ ones per row, $2p$ or $2p+1$ ones per column and inner product $2p$. So if we let $f$ be a $1\times p(2p+1)$ matrix of ones

$$\begin{bmatrix} f & 0 \\ C & A \end{bmatrix}$$

is a BIBD($2p^2+p+1$, $p(2p^2+p+1)$, $p(2p+1)$, $2p+1$, $2p$). □

**COROLLARY 4.** *Let p be any prime power and q any integer. Then there exists a* PBIBD($2p^2$, $p(p+q)(2p-1)$, $(p+q)(2p-1)$, $2p$, $\lambda_1=q(2p-1)$, $\lambda_2=2p-1+q$).

*Proof:* As in the proof of Theorem 3, we use the GH(2p, EA(p)) to first form a

GBRD(2p, p(2p-1), p(2p-1), 2p, p(2p-1); EA(p)).

This then yields a

GDD($2p^2$, $p^2(2p-1)$, $p(2p-1)$, $2p$, $\lambda_1=0$, $\lambda_2=2p-1$),

A. Form C as before from a

BIBD(2p, qp(2p-1), q(2p-1), 2, q).

Then [C:A] is a

PBIBD($2p^2$, p(p+q)(2p-1), (p+q)(2p-1), 2p, $\lambda_1=q(2p-1)$, $\lambda_2=2p-1+q$). □

**Example 3.** A GH(6,EA(3)) exists so there is a GBRD(6,10,10,6,10;EA(3)). This can be used with a BIBD(7,21,6,2,1) to form a BIBD(22,66,21,7,6).

**Example 4.** A GH(18,EA(9)) exists, so there is a GBRD(18, 153, 153, 18, 153; EA(9)). This is used with a BIBD(19, 171, 18, 2, 1) to form a BIBD(172, $9 \cdot 172$, 171, 19, 18).

All the following constructions can be obtained by a similar, slightly modified, technique.

**THEOREM 5.** *Suppose there exists a* GH(tg,G), $g = |G|$ *odd. Further suppose that there exists a* BIBD($tg+1$, $s(tg+1)$, $ts$, $t$, $\lambda$). *Then there exists a* BIBD($tg^2+g+1$, $\alpha s(tg^2+g+1)$, $\alpha s(tg+1)$, $tg+1$, $\alpha st$) *where* $s = \lambda g/(t-1)$ *is an integer and* $2\alpha\lambda(tg-t+1) = \beta t(tg-1)(t-1)$ *for some* $\alpha$ *and* $\beta$. *In particular, if* $\alpha = \binom{tg}{2}$ *and* $\beta = tg - t + 1$, *there is a*

$$\mathrm{BIBD}\left(tg^2+g+1,\ s(tg^2+g+1)\binom{tg}{2},\ s(tg+1)\binom{tg}{2},\ tg+1,\ st\binom{tg}{2}\right).$$

*Proof:* From theorem 1, there exists a GBRD($tg$, $\binom{tg}{2}$, $\binom{tg}{2}$, $tg$, $\binom{tg}{2}$; $G$). We replace each element of G by its $p\times p$ permutation matrix to form a (0,1) matrix E. Further, let e be the $1\times\binom{tg}{2}$ matrix of ones. Then,

$$B = \begin{bmatrix} I_g \times e \\ E \end{bmatrix}$$

is a GDD($g(tg+1)$, $g\binom{tg}{2}$, $\binom{tg}{2}$, $tg+1$, $\lambda_1 = 0$, $\lambda_2 = \frac{1}{2}t(tg-1)$, $m = g$, $n = tg+1$). We now replace each 0 and 1 of the

$$\mathrm{BIBD}(tg+1,\ \lambda g(tg+1)/(t-1),\ \lambda tg/(t-1),\ t,\ \lambda)$$

by the $g\times 1$ matrix of zeros and ones respectively to form a GDD($g(tg+1)$, $\lambda g(tg+1)/(t-1)$, $\lambda tg/(t-1)$, $tg$, $\lambda_1 = \lambda tg/(t-1)$, $\lambda_2 = \lambda$, $m = g$, $n = tg+1$), A.

We now form the following (0,1) matrix:

$$C = \begin{bmatrix} 11\cdots 11 & | & 00\cdots 00 \\ \alpha copies A & | & \beta copies B \end{bmatrix}$$

The first row of C has $\alpha\lambda g(tg+1)/(t-1)$ ones and has intersection $\alpha\lambda tg/(t-1)$ with the other rows of C.

Every other row of C has $\alpha\lambda tg/(t-1) + \beta\frac{1}{2}tg(tg-1)$ ones. So we require

$$\alpha\lambda g(tg+1)/(t-1) = \alpha\lambda tg/(t-1) + \beta\tfrac{1}{2}tg(tg-1)$$

or

$$\alpha\lambda = \beta\tfrac{1}{2}t(tg-1)(t-1)/(tg-t+1) \qquad (1)$$

The intersection numbers for the rows are required to be equal, so we need

$$\alpha\lambda tg \,/\, (t-1) = \alpha\lambda tg \,/\, (t-1) + \beta \cdot 0 = \alpha\lambda + \beta \tfrac{1}{2} t(tg-1)$$

or, as in (1)

$$\alpha\lambda = \beta \tfrac{1}{2} t(tg-1)(t-1) \,/\, (tg-t+1).$$

Thus C is a
BIBD($tg^2+g+1$, $\alpha\lambda g\,(tg^2+g+1)/(t-1)$, $\alpha\lambda g\,(tg+1)/(t-1)$, $tg+1$, $\alpha\lambda tg\,/(t-1)$).
Where $\lambda g \,/\, (t-1) = s$ an integer, and a possible solution for $\alpha$ and $\beta$ is $\alpha = \left\lceil \frac{tg}{2} \right\rceil$, $\beta = s(tg-t+1)$. That is, C is a

$$\mathrm{BIBD}(tg^2+g+1,\ s(tg^2+g+1)\left\lceil \tfrac{tg}{2} \right\rceil, s(tg+1)\left\lceil \tfrac{tg}{2} \right\rceil,\ tg+1, st\left\lceil \tfrac{tg}{2} \right\rceil). \qquad \square$$

**COROLLARY 6.** *Let $g$ and $g$-1 be prime powers, $g$ odd. If there exists a* BIBD($g^2-g+1$, $g(g^2-g+1)$, $g(g-1)$, $g-1$, $g-2$) *then there exists a* BIBD($g^3-g^2+g+1$, $\alpha g(g^3-g^2+g+1)$, $\alpha g(g^2-g+1)$, $g^2-g+1$, $\alpha g(g-1)$), *where* $2\alpha(g^2-2g+2) = \beta(g-1)(g^2-g-1)$ *has an integer solution.*

*Proof:* By a Theorem of Rajkundlia (1978) and Seberry (1981), a GH(g(g-1);EA(g)) always exists in these cases. □

**Remark.** The BIBD obtained would be a multiple of an SBIBD($g^3-g^2+g+1, g^2-g+1, g-1$) which theoretically, can never exist, as $g^3-g^2+g+1$ is even and $k-\lambda = g^2-2g+2$ is not a square.

**Example 5.** Let $g=5$. There exists a BIBD(21,105,20,4,3). Hence, there exists a BIBD(106, 38·5·106, 38·5·21, 21, 38·5·4), $\alpha = 38$. This is a multiple of the SBIBD(106,21,4) which is non-existent.

**COROLLARY 7.** *Let $g$ be an odd prime power. Let $\alpha = 2(4g-1)$. Then there is a* BIBD($4g^2+g+1$, $2g^2(4g^2+g+1)(4g-1)$, $2g^2(4g+1)(4g-1)$, $4g+1$, $8g^2(4g-1)$).

*Proof:* Dawson (1985) has shown a GH(4g,(EA(g)) always exists. Also, the required BIBD($4g+1$, $g(4g+1)$, $4g$, 4, 3) always exists and so, with $\alpha = 2(4g-1)$, $\beta = 4g-3$ in Theorem 5, we get the result. □

**Remark.** This would be a multiple of the SBIBD($4g^2+g+1$, $4g+1$, 4) but this can only exist ( since $4g^2+g+1$ is even ) if $k-\lambda = 4g-3$ is a square.

**Example 6.** Let $g=9$. Then $\alpha = 70$ and a BIBD(334,70·9·334,70·9·37,37,36·70) exists.

**COROLLARY 8.** *Let $g = 3^h$. Then there exists*
BIBD($4g^2+g+1$, $\alpha\lambda g(4g^2+g+1)/3$, $\alpha\lambda g(4g+1)/3$, $4g+1$, $4\alpha\lambda g/3$)
*where* $2\alpha\lambda(4g-3) = 12\beta(4g-1)$ *for some $\alpha$ and $\beta$. In particular, if* $\alpha\lambda = 2(4g-1)$

*and* $\beta = (4g-3)/3$, *there is a*
BIBD($4g^2+g+1$, $2g(4g-1)(4g^2+g+1)/3$, $2g(4g+1)(4g-1)/3$, $4g+1$, $8g(4g-1)/3$).

*Proof:* We again use the GH(4g, EA(g)) found by Dawson (1985). We note that a BIBD($4g+1$, $\lambda g(4g+1)/3$, $4\lambda g/3$, 4, $\lambda$) exists for all $\lambda$. We use these in Theorem 5 to get the result. □

**Remark.** The constructed designs are also multiples of an SBIBD($4g^2 + g + 1$, $4g + 1$, 4) which never exists as $4g^2 + g + 1$ is even and $k - \lambda = 4g - 3$ is never a square for $g = 3^h$, $h>1$.

**COROLLARY 9.** *Let $p$ be an odd prime power. Suppose there exists a* BIBD($p^i + 1$, $qp^j(p^i + 1)$, $qp^i$, $p^{i-j}$, $q(p^{i-j} - 1)$) *where $i \geq j$, and $q$ are integers. Then there exists a*

$$\text{BIBD}(p^{i+j} + p^j + 1,\ \alpha qp^j(p^{i+j} + p^j + 1),\ \alpha qp^j(p^i + 1),\ p^i + 1,\ \alpha qp^i))$$

*where* $2\alpha q(p^i - p^{i-j} + 1) = \beta p^{i-j}(p^i - 1)$, *there is a*

$$\text{BIBD}(p^{i+j} + p^j + 1,\ p^i(p^i - 1)(p^{i+j} + p^j + 1),\ p^i(p^{2i} - 1),\ p^i + 1,\ p^{2i-j}(p^i - 1)).$$

*Proof:* Use the GH($p^i$, EA($p^j$), $i > j$ given by Drake (1979) or Butson (1963). □

**Remark.** This would be a multiple of the SBIBD($p^{i+j}+p^j+1$, $p^i+1$, $p^{i-j}$). Since $p^{i+j}+p^j+1$ is odd, in order for this to exist, the diophantine equation

$$z^2 = (p^i - p^{i-j} + 1)x^2 + (-1)^{\frac{1}{2}(p^i+1)}p^{i-j}y^2$$

must have a solution in the integers for x, y, z not all zero.

**Example 7.** Let i = 2, j = 1, q = 1 and p = 5. A BIBD(26, 130, 25, 5, 4) exists. Hence a BIBD(131, 600·131, 600·26, 26, 600·5) exists.

## 4. USING GENERALIZED WEIGHING MATRICES IN THE CONSTRUCTION

As noted in Seberry (1979), and Geramita and Seberry (1979), infinite families of GW matrices are known.

**THEOREM 10.** *Let $p^r$ be a prime power and $q \mid p^r-1$, $q$ odd. Then there exists a*

$$\text{GBRD}(p^r+1, \tfrac{1}{2}p^r(p^r+1), \tfrac{1}{2}p^r(p^r-1), p^r-1, \tfrac{1}{2}(p^r-1)(p^r-2); Z_q)$$

*and $B$, a GDD with parameters*

$$(q(p^r+1), \tfrac{1}{2}qp^r(p^r+1), \tfrac{1}{2}p^r(p^r-1), p^r-1, \lambda_1=0, \lambda_2=\tfrac{1}{2}(p^r-1)(p^r-2)/q, m=q, n=p^r+1).$$

*Hence if, $q = p^r - 1$, there exists a*

$$BIBD\,(p^{2r}-1, \tfrac{1}{2}(p^{2r}-2)(p^r+1), \tfrac{1}{2}(p^{2r}-2), p^r-1, \tfrac{1}{2}(p^r-2)).$$

*If $q \mid p^r - 1$, $q \neq p^r - 1$ and there exists a BIBD ($p^r + 1$, $b$, $\rho$, $(p^r - 1)/q$, $\lambda$), $A$, where*

$\lambda q p^r = \rho(p^r - q - 1)$. *Using A to form a*

$$GDD\,(q\,(p^r + 1),\ qb,\ \rho,\ p^r - 1, \lambda_1 = r, \lambda_2 = \lambda)$$

*then*

$$[\alpha \text{ copies of } A : \beta \text{ copies of } B\ ]$$

*where* $2q\alpha(\rho - \lambda) = \beta(p^r - 1)(p^r - 2)$ *gives a* $BIBD\,(q\,(p^r + 1),\ B,\ R,\ p^r - 1, \alpha\rho)$.

*Proof:* We note first that a $GW(p^r{+}1, p^r, p^r{-}1; Z_q)$ exists for all $p^r$. The proof, then, is identical to the first part of the proof of Theorem 3. □

**Example 8.** A $GW(17, 16, 15; Z_q)$, $q = 15$, 5 and 3 exists. This gives a BIBD(15·17, 127·17, 127, 15, 7). Also, we have GDD(5·17, 40·17, 120, 15, $\lambda_1 = 0$, $\lambda_2 = 21$, m = 5 , n = 17) and a GDD(3·17, 24·17, 120, 15, $\lambda_1 = 0$, $\lambda_2 = 35$, m = 3, n = 17). Since BIBD(17, 8·17, 24, 3, 3) and BIBD(17, 4·17, 20, 5, 5) exist, we have a BIBD(85, 34·24, 6·24, 15, 24) with q = 5, r = 24, $\lambda = 3$, $\alpha = \beta = 1$ and a BIBD(51, 1700, 500, 15, 140) with q = 3, r = 20, $\lambda = 5$, $\alpha = 7$, and $\beta = 3$.

**Example 9.**

We note that there exists a

$$GW((p^{n+1} - 1)/(p - 1), p^n; Z_q)$$

for all $q \mid p^n(p - 1)$. So we can choose $q$ odd and proceed as in the previous theorem. We do not give full results but note some examples: the GW(21, 16; $Z_3$) gives a GBRD(21, 12, 66; $Z_3$) and a BIBD(63, 12, 22·60), the GW(31, 25; $Z_5$) gives a GBRD(31, 20, 190; $Z_5$) and a BIBD(155, 20, 19·20), and the GW(85, 64; $Z_3$) gives a GBRD(85, 48, 24·47; $Z_3$) and a BIBD(255, 24, 94·336).

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